Thermowell vibrations

1. Introduction

Most thermowell design calculations consist of using the ASME PTC 19.3 - 1974 (1986) vibration and stress calculation method. The method unfortunately has many drawbacks, first it is based on certain thermowell dimensions, it is obvious that increasing the diameter of the thermowell raises the resonant frequency, however using ASME, you can’t take advantage of this and therefore cannot design thermowells correctly.
Also the phenomenon also applies to sample quills (often parallel) again the method cannot be applied in this case when in fact it should be possible to apply it.

Refer to figure 1 for dimensions.

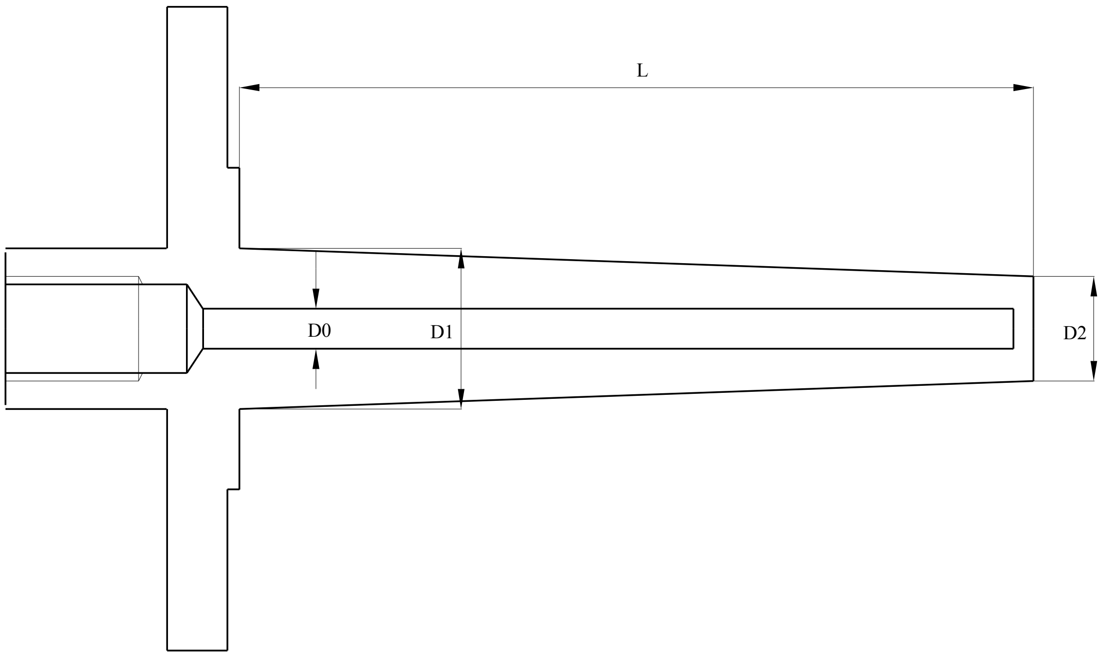

Fig 1

2.0 Thermowell Calculations to ASME PTC 19.3 - 1974

Natural Freq of Probe: $f_n = \frac{K_f}{L^2}\sqrt{\frac{E}{\gamma}}$ Hz (K_f from table 1.4 below) (1)

Well dimensions in inches (Table 1.3) [Note only the minimum values used to yield minimum frequency, notation is consistent with Fig 1 above]:

Bore d (in)	1/4	3/8	9/16	11/16	7/8
Dimensions					
D1 (min)	13/16	15/16	1 1/8	1 1/4	1 7/16
D2 (min)	5/8	3/4	15/16	1 1/16	1 1/4
d (min)	0.254	0.379	0.566	0.691	0.879

Values of Kf (Table 1.4)

Bore d ->	**0.25**	**0.375**	**0.5625**	**0.6875**	**0.875**
Length L					
2.5 Inches	2.06	2.42	2.97	3.32	3.84
4.5 Inches	2.07	2.45	3.01	3.39	3.96
7.5 Inches	2.08	2.46	3.05	3.44	4.03
10.5 Inches	2.09	2.47	3.06	3.46	4.06
16 Inches	2.09	2.47	3.07	3.47	4.08
24 Inches	2.09	2.47	3.07	3.48	4.09

E is Young's modulus in PSI for the material (29 10^6 for steel), γ is the density in Lbs/cu.in (0.29 for steel), L is the length in inches.

First, notice how the taper is not taken into account by the formula, Kf depends uniquely on the bore, d, and the length, L. This is non-sensical unless the design applies to a single, unique, diameter well. How could thermowells with a 7.5in insertion length and a 1/4" bore all vibrate at the same frequency, common sense suggests that a 30mm one should vibrate at a higher frequency than a 22 mm one.
Obviously it means the formula has been based on the minimum diameter and increasing this one cannot lead on an improvement in the design since the formula does not take it into account.

3.0 <u>Analysis of a cylindrical (parallel) thermowell (or sample quill)</u>

It is relatively easy to work out exactly, the fundamental natural resonance of a cantilevered cylinder with a concentric bore. (ie parallel thermowell).

The moment of inertia of the thermowell is everywhere the same and equal to:

$$I=\frac{\pi\left(D^4-d^4\right)}{64}$$ Eq. 3.1 where D is the outside diameter, and d the bore diameter.

The cross section is:

$$A=\frac{\pi\left(D^2-d^2\right)}{4}$$ Eq. 3.2

The general motion of a beam of uniform cross section in free motion is:

$$EI\frac{\partial^4 y}{\partial x^4}+\rho A\frac{\partial^2 y}{\partial t^2}=0$$ Eq. 3.3

Where y is the deflection of the beam (as a function of x and t)

Because we are looking at a single harmonic function, y(x,t) can be written as:

$y(x,t) = y(x) \cos(\omega t)$ where $\omega = 2\pi fn$ (fn = natural frequency)

The equation to be solved is now:

$$\frac{d^4 y}{d x^4} - \rho A \frac{\omega^2}{EI} y = 0 \quad \text{Eq. 3.4}$$

If we call $z^4 = \frac{\rho A \omega^2}{EI}$

The generalised solution of the y(x) equation is:

$$y = A_1 \sinh(zx) + A_2 \cosh(zx) + A_3 \sin(zx) + A_4 \cos(zx) \quad \text{Eq. 3.5}$$

Where the A's are constants which depend only on the boundary conditions.

In the case of the thermowell, we have certain boundary conditions to satisfy:

1) at x = 0 y is always 0 as the thermowell at this point is fixed
   this implies $A_2 = -A_4$ Eq. 3.6

2) at x = 0 , the slope is always 0 (dy/dx = 0) as the thermowell is not free to rotate,
   this implies $A_1 = -A_3$ Eq. 3.7

3) at x = L, the bending moment is always 0 hence $\frac{d^2 y}{d x^2} = 0$,
   this implies, taking into account the relations obtained at 1) and 2):

$$A_2 = -A_1 \frac{[\sinh(zL) + \sin(zL)]}{[\cosh(zL) + \cos(zL)]} \quad \text{Eq. 3.8}$$

4) at x = L, the shear force is also always 0 hence $\frac{d^3 y}{d x^3} = 0$, taking into account all
   previous relations we finally get:

$$A_1 \frac{[1 + \cos(zL)\cosh(zL)]}{[\cos(zL) + \cosh(zL)]} = 0 \quad \text{Eq. 3.9}$$

For this to be true no regardless of A_1 is we must have $[1 + \cos(zL)\cosh(zL)] = 0$

an infinity of values satisfy this equation, but the lowest value is zL = 1.875104

This corresponds to $\omega = \frac{3.516015}{L^2} \sqrt{\frac{EI}{\rho A}}$ or in frequency:

$$f_n = \frac{0.55959}{L^2}\sqrt{\frac{EI}{\rho A}}$$ Eq. 3.10

If we call $\delta = \frac{d}{D}$ we can write I and A as a function of D and δ

$$I = \frac{\pi D^4}{64}\left(1-\delta^4\right)$$
$$A = \frac{\pi D^2}{4}\left(1-\delta^2\right)$$

And the frequency formula becomes:

$$f_n = \frac{0.139898 D}{L^2}\sqrt{\frac{E\left(1+\delta^2\right)}{\rho}}$$ eq. 3.11

This formula is exact for parallel thermowells:
In metric units, E is in Pa, L and D in metres, ρ is in kg/m3.

4.0 Tapered Thermowells natural frequency

It is also possible to calculate approximately the natural resonant frequency of a tapered thermowell based on the Rayleigh method which consists of assuming a deflected shape for the thermowell which is reasonable (ie consistent with the boundary conditions) then calculating the potential energy at maximum deflection (velocity everywhere equal to 0) and the maximum kinetic energy (deflection everywhere equal to 0) and equalling the two since by the principle of conservation of energy, the sum potential + kinetic =constant .

Notation:

$k = D_2/D_1$
$\delta = D_0/D_1$.
E = Young's Modulus ($2 \,.\, 10^{11}$ Pa for steel and stainless steel).
ρ = Density of the material (7800 kg/m3 for steel).
L, D_0, D_1, D_2 are in metres

The stiffness is a function of the inertia of the cross section. Because the cross section varies along the length, so does the inertia (2nd moment of area). The inertia is nominated as I(x) where x is the distance measured from the inner face of the flange (or end of the thread for screwed thermowells).

$$I_{(x)} = \frac{\pi}{64}\left(D_{(x)}^4 - D_0^4\right)$$ Eq 4.1 where D(x) is the diameter at x.

$$D_{(x)} = D_1\left(1 - \frac{x}{L}\left(1-k\right)\right)$$ Eq. 4.2

hence we get for I(x):

$$I_{(x)}=\frac{\pi . D_1^4}{64}\left[\left(1-\frac{x}{L}(1-k)\right)^4-d^4\right] \quad \text{Eq. 4.3}$$

where $k=\frac{D_2}{D_1}$ and $\delta=\frac{D_0}{D_1}$

The simplest deflected shape which we can use is :

$$y=\lambda\left(\frac{x^4}{L^4}-4\frac{x^3}{L^3}+6\frac{x^2}{L^2}\right).\sin(\omega t) \quad \text{Eq. 4.4}$$

as it is consistent with the cantilevered condition at x = 0 & x=L since it is the deflected shape of a cantilever of uniform section subjected to a uniform load (λ is an arbitrary constant).

The instantaneous speed is $\frac{\partial y}{\partial t}=\omega\lambda\left(\frac{x^4}{L^4}-4\frac{x^3}{L^3}+6\frac{x^2}{L^2}\right).\cos(\omega t)$ Eq. 4.5

the maximum kinetic energy occurs when cos(ωt)=1 and is given by:

$$U_k=\frac{1}{2}\int_0^L\left[\rho\frac{\pi\left(D_{(x)}^2-D_0^2\right)}{4}\right].\left[\frac{\partial y}{\partial t}\right]_{\mathrm{MAX}}^2 dx \quad \text{Eq.4.6}$$

or

$$U_k=\frac{\pi}{8}\int_0^L \rho\left(D_{(x)}^2-D_0^2\right).\lambda^2\omega^2\left(\frac{x^4}{L^4}-4\frac{x^3}{L^3}+6\frac{x^2}{L^2}\right)^2.dx \quad \text{Eq. 4.7}$$

$$U_k=\frac{\pi\,\omega^2\lambda^2\rho\,L D_1^2}{27720}\left[5353k^2+2142k+513-8008d^2\right] \quad \text{Eq. 4.8}$$

The internal energy in the thermowell at maximum deflection is:

$$U_i=\frac{1}{2}\int_0^L\frac{M_{(x)}^2}{E_y}.I_{(x)}.dx \quad \text{Eq. 4.9}$$

where M(x) is the bending moment at x

The bending moment, M(x), is a function of the stiffness and curvature of the deflected thermowell:

$$M_{(x)}=E_y.I_{(x)}.\frac{\partial^2 y}{\partial x^2} \quad \text{Eq. 4.10}$$

Since we assumed the shape, we have :

$$\left[\frac{\partial^2 y}{\partial x^2}\right]_{MAX} = \frac{12\lambda}{L^2}\left(\frac{x^2}{L^2} - 2\frac{x}{L} + 1\right) \qquad \text{Eq. 4.11}$$

We get for the internal energy :

$$U_i = \frac{144\lambda^2 E_y}{2L^4}\int_0^L I_{(x)}\left(\frac{x^2}{L^2} - 2\frac{x}{L} + 1\right)^2 dx \qquad \text{Eq. 4.12}$$

Replacing I(x) by its expression in equation 2.3 and integrating we finally obtain:

$$U_i = \frac{\lambda^2 . E_y . \pi . D_1^4 \left[k^4 + 5.k^3 + 15.k^2 + 35.k + 70 - 126.\delta^4\right]}{560.L^3} \qquad \text{Eq. 4.13}$$

and finally we get by equating Ui and Uk since according to the principle of conservation of energy, when the kinetic energy is maximum, the potential (internal) energy is zero and vice-versa.

$$\omega^2 = \frac{99.E_y.D_1^2}{2.L^4.\rho}\left[\frac{k^4 + 5.k^3 + 15.k^2 + 35.k + 70 - 126.\delta^4}{5353.k^2 + 2142.k + 513 - 8008.\delta^2}\right] \qquad \text{Eq. 4.14}$$

The natural frequency is to a close approximation:

$$f_n = \frac{\omega}{2\pi} = 1.1197\frac{D_1}{L^2}\sqrt{\frac{E_y}{\rho}\left[\frac{k^4 + 5.k^3 + 15.k^2 + 35.k + 70 - 126.\delta^4}{5353.k^2 + 2142.k + 513 - 8008.\delta^2}\right]} \qquad \text{Eq. 4.15}$$

5.0 Comparison between tapered formula and parallel formula

Because the parameter k in the tapered equation 4.15 can be made equal to 1 (ie $D_1 = D_2$) the two formulas Eq.4.15 and Eq 3.11 can be compared.

Example:
Let's compare a steel cylindrical (ie parallel) thermowell 7.5 inches long and 25.4mm in diameter, and two bores : a 0.25 in and a 0.875 in

Bore (in)	0.25	0.875
Tapered Eq 4.15	515.8 Hz	664.9 Hz
Exact Eq. 3.11	513.77 Hz	662.30Hz

This gives us an indication of the accuracy of the approximate method, we can expect errors less than 1% from the correct values.

6.0 Comparison between ASME and Equation 4.15

Let's choose 2 thermowells based on ASME minimum dimension table say one with a small bore of 1/4" and the other with a bore of 7/8".

And we will compare the two types of tapered wells over the full range of lengths. The calculations are based on steel thermowells E = 200 000 MPa, ρ=7800 kg/m3.
The two tables below give the frequencies in Hz for the two selected bores.

	bore =0.25"					
Length (in)	2.5	4.5	7.5	10.5	16	24
ASME fn (Hz)	3296.0	1022.2	369.8	189.57	81.64	36.28
Eq 4.15 fn (Hz)	4456	1375.6	495.2	252.7	108.8	48.4

	bore =0.875"					
Length (in)	2.5	4.5	7.5	10.5	16	24
ASME fn (Hz)	6144	1956	716	368	159	71
Eq 4.15 fn (Hz)	8680	2679	964	492	212	94.2

As can be seen, the ASME value underestimate the frequency by about 30% leading to an extremely conservative design. In addition, the minimum values for the dimensions have been used, using the maximum values would have resulted in an even more conservative value (since ASME takes no account of the taper in its formula).

But my main objection to the ASME formula is that the bore and the external diameter are linked intrinsically. Eq 4.15 allows you to vary bore, taper, and external size independently hence it is a much more versatile formula. And by comparison with the cylindrical thermowell whose formula is exact, we know that the order of error induced by the approximation is less than 1%.

7.0 Conclusion

I recommend that the ASME method no longer be used, and the above formula (Eq 4.15) used instead, limiting the Strouhal frequency to 75% of the resonant value.

Part II will consider stresses and fatigue.